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Mathematical formulation for the propagation of sound through a turbulent jet

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SUMMARY

The sound propagation through a nonuniform turbulent jet flow field is studied by means of a system of linearized equations governing the acoustic variables. These equations depend on the fluctuating flow-field variables which can be prescribed by experimental results. It is shown that the correlations of the acoustic variables depend throughout the flow field on the space-time correlation of the turbulent velocities and on the mean flow variables and their gradients.

1. Introduction

Previous work by Liu and Maestrello [1] on sound propagation through a jet flow field containing gradients of mean velocities and pressure indicates that in order to better estimate the directional redistribution of acoustic energy the fluctuations in pressure and velocities have to be considered. These fluctuations, caused by the random inhomogeneity of the turbulent field in the jet, lead to a random variation of the refractive index, as previous papers by Lighthill [2], Kraichnan [3], and Batchelor [4] indicate. This causes a redistribution of acoustic energy, in addition to and in association with refraction and convection effects.

Historically, the scattering problem has been notoriously difficult. In a jet the turbulent field is nonisentropic, and the turbulent and acoustic intensities are not weak, as required for Born's approximation. In addition, one has to solve for multiple scattering effects due to the nonhomogeneous volume of turbulence in the jet. Recently, Crow [5] and Howe [6] extended the knowledge in this field. Crow proposed a visco-elastic theory applicable to fields whose wavelengths greatly exceed the correlation scale of the turbulent motions, without accounting for multiple scattering effects. On the other hand, Howe accounted for

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multiple scattering under specialized conditions. He considered the case where the fluctuations of the media are caused by variations in temperature fluctuations in space and time producing a random fluctuation in the speed of sound. In general, however, the problems of turbulent scattering were confined to problems involving inhomogeneities in space alone.

The present work examines the problem of scattered sound as a propagation phenomenon. Information about the turbulent and mean flow variables are to be deduced from measurements of spatial, temporal, and spatial-temporal variations through the jet flow field. The governing equations for the mean and fluctuating acoustic variables are derived. The former are more general than those used by Schubert [7] or Liu and Maestrello [1] since no assumptions are imposed to reduce the system of equations to a single convective wave equation for the pressure. The equations governing the fluctuating acoustic variables are recast in terms of correlations of both fluctuating flow and acoustic variables.

The necessary correlations of the turbulent flow field are to be deduced from experimental data which has been obtained by Maestrello, et al. [8].

2. Governing equations

The basic governing equations that describe the propagation of sound through a jet flow field are the inviscid equations of motion. The independent variables are the cylindrical coordinates $\hat{x}$, $\hat{r}$, and θ (Fig. 1) and the time $\hat{t}$. The dependent variables are the velocity components $(\hat{u}, \hat{v}, \hat{w})$ in the $(\hat{x}, \hat{r}, \theta)$ direction, the density $\hat{\rho}$, and the pressure $\hat{p}$. All variables are nondimensionalized with respect to jet exit conditions:

$$(u, v, w) = (\hat{u}/\bar{a}_E, \hat{v}/\bar{a}_E, \hat{w}/\bar{a}_E),\ \rho = \hat{\rho}/\bar{\rho}_E,\ p = \hat{p}/\bar{\rho}_E\bar{a}_E^2, \tag{1}$$

$$(x, r, \theta) = (\hat{x}/d_0, \hat{r}/d_0, \theta),\ t = \bar{a}_E\hat{t}/d_0,$$

where d_0 is the exit diameter, $\bar{\rho}_E$ the mean jet exit density and $\bar{a}_E$ the mean jet exit speed of sound.

In terms of the nondimensional variables on the left-hand side of Equation (1), the governing equations are:

$$\frac{\partial\phi}{\partial t} + A_1\frac{\partial\phi}{\partial x} + B_1\frac{\partial\phi}{\partial r} + C_1\frac{1}{r}\frac{\partial\phi}{\partial\theta} + D_1\phi = 0, \tag{2}$$

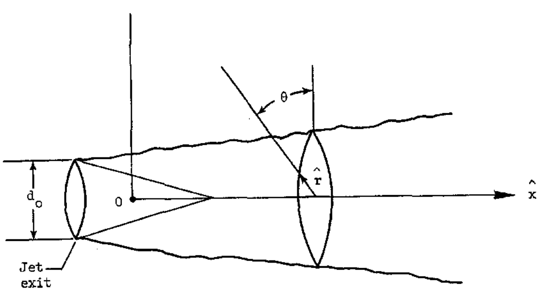

Figure 1.

where

$$\phi^T = (\rho, u, v, w, p),$$

$$A_1 = \begin{pmatrix} u & \rho & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 1/\rho \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & 0 & u \end{pmatrix}, \quad B_1 = \begin{pmatrix} v & 0 & \rho & 0 & 0 \\ 0 & v & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 1/\rho \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & \gamma p & 0 & v \end{pmatrix},$$

$$C_1 = \begin{pmatrix} w & 0 & 0 & \rho & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & w & 1/\rho \\ 0 & 0 & 0 & \gamma p & w \end{pmatrix}, \quad D_1 = \frac{1}{r}\begin{pmatrix} v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -w & 0 \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & \gamma v \end{pmatrix}. \tag{3}$$

Let $\bar{\phi}$ be due to the mean flow and ϕ' be due to the fluctuating flow due to turbulence. The word mean is used in the usual sense, that is, time average over the time scale of the turbulence. Thus the mean of a fluctuating flow variable vanishes. The perturbation due to the addition of a weak acoustic source is denoted by $\tilde{\phi}$.

The dependent variables are then written as the sum of three components, for example,

$$\phi = \bar{\phi} + \phi' + \tilde{\phi}. \tag{4}$$

The nonlinear governing equation (2) can be linearized due to the assumption

$$|\tilde{\phi}|/|\bar{\phi} + \phi'| = O(\varepsilon) \ll 1. \tag{5}$$

The leading term gives the governing equation for $(\bar{\phi} + \phi')$ which is the same as that for ϕ, that is, Equation (2). Note that the mean of this equation yields the standard equation for ϕ with ϕ' appearing in the Reynolds stress term. The next order equation, which is $O(|\tilde{\phi}|/|\bar{\phi} + \phi'|)$, governs $\tilde{\phi}$ and is given by:

$$\frac{\partial\tilde{\phi}}{\partial t} + A_2\frac{\partial\tilde{\phi}}{\partial x} + B_2\frac{\partial\tilde{\phi}}{\partial r} + C_2\frac{1}{r}\frac{\partial\tilde{\phi}}{\partial\theta} + A_3\frac{\partial(\bar{\phi} + \phi')}{\partial x} + B_3\frac{\partial(\bar{\phi} + \phi')}{\partial r}$$
$$+ C_3\frac{\partial(\bar{\phi} + \phi')}{r\partial\theta} + D_3(\bar{\phi} + \phi') = 0, \tag{6}$$

where A_2, and so forth, are the matrices (3) with the elements replaced by the corresponding component of $(\bar{\phi} + \phi')$, and A_3, and so forth, are the matrices (3) with the elements replaced by the corresponding component of $\tilde{\phi}$ with the exception that $1/\rho$ is replaced by $-\tilde{\rho}/(\bar{\rho} + \rho')^2$.

The fluctuating flow field, described by ϕ', is, of course, time dependent. However, the time scale of the turbulence is different from that of the source. A simple comparison of the orders of magnitude of the frequencies evidences the different time scales. The scale of the fluctuating flow frequency is given by $\omega_T = \bar{u}/l$ where l is the length scale of the turbulence in the direction of $\bar{u}$, and $\bar{u}$ is the largest component of the mean flow velocity. The scale of the frequencies of the periodic acoustic sources is given by $\omega = \bar{a}/d_0$. Since the jet is subsonic $\bar{u} < \bar{a}$. In general $l > d_0$. Therefore $\omega_T \ll \omega$, so that in the scale of the acoustic sources, the turbulence appears to be steady. If we denote σ as the time scale of the turbulence, then for an axially symmetric jet

$$\phi' = \phi'(x, r, \sigma)$$

and therefore is not a function of t.

Equation (6) is linear in $\tilde{\phi}(x, r, \theta, \sigma, t)$ with coefficients independent of t. Therefore, $\tilde{\phi}$ can be represented as:

$$\tilde{\phi} = \psi(x, r, \theta, \sigma)e^{i\omega t}.$$

Substituting into Equation (6) and factoring $e^{i\omega t}$ from all terms results in the governing equation for ψ which is given by Equation (6) with $\tilde{\phi}$ replaced by ψ and $\partial/\partial t$ replaced by $i\omega + \partial/\partial\sigma$.

A further simplification can be effected if $|\phi'/\bar{\phi}| = O(\delta)$ is small. However, the amplitude of the point source is much smaller than the amplitude of the turbulent flow fluctuations, that is,

$$\varepsilon \ll \delta.$$

Therefore, the point source will not change the characteristics of the turbulent flow field. Equation (6) has an error of $O(\varepsilon^2)$, while Equation (2), the equation governing $(\bar{\phi} + \phi')$, includes all terms independent of ε. The acoustic amplitude variable is now written as

$$\psi(x, r, \theta, \sigma) = \bar{\psi}(x, r, \theta) + \psi'(x, r, \theta, \sigma) \tag{7}$$

where $\bar{\psi}$ denotes the mean of ψ over the turbulent time scale and the mean of ψ' vanishes. $\bar{\psi}$ will represent the amplitude of the acoustic response due to the mean flow field $\bar{\phi}$, and ψ' will represent the effects of scattering due to turbulence. Consistent with the assumption of the order of magnitude of $|\phi'/\bar{\phi}|$, it can be assumed that $|\psi'/\bar{\psi}| = O(\delta)$.

Equation (7) is now substituted into Equation (6) for ψ. Terms of $O(\varepsilon\delta^2)$ or higher are neglected. This means that products of ψ' with anything except $\bar{\phi}$ and products of ϕ' with anything except $\bar{\psi}$ are neglected. Note that products of ϕ' with $\bar{\phi}$ or ϕ' are already included in the leading order equation. The resulting equation will involve products of $\bar{\psi}$ and $\bar{\phi}$, which are $O(\varepsilon)$, products of ψ' and $\bar{\phi}$ which are $O(\varepsilon\delta)$, and products of ϕ' and $\bar{\psi}$ which are also $O(\varepsilon\delta)$. The equation is given by:

$$\frac{\partial\psi'}{\partial\sigma} + P(\bar{\psi} + \psi') + N(\phi') = 0 \tag{8}$$

where the differential operator P has coefficients which depend on the components of $\bar{\phi}$, and the differential operator N has coefficients which depend on $\bar{\phi}$ and $\bar{\psi}$, the latter appearing linearly. Appendix I contains a detailed description of P and N. It is important to notice that P and N are independent of fluctuating variables so that Equation (8) is linear in the fluctuating variable of the flow, ϕ', and of the acoustic response, ψ'. Therefore, if we take the mean over the turbulent time scale of Equation (8), the result is the equation governing $\bar{\psi}$:

$$P\bar{\psi} = 0. \tag{9}$$

This is the equation governing the problem considered by Schubert [7] and Liu and Maestrello [1], among others, and which is here placed in the proper context within a scattering problem. Substitution of Equation (9) into Equation (8) results in the governing equation for ψ':

$$P\psi' = -N\phi' \tag{10}$$

where the operator N depends on the solution of Equation (9), and where the $\partial/\partial\sigma$ term is of higher order when compared to the $i\omega$ term of P.

3. Discussion of governing equations

In order to solve Equation (9) for $\bar{\psi}$, the solution for $\bar{\phi}$ is needed to evaluate the coefficients in the operator P. To avoid solving the nonlinear equations governing $\bar{\phi}$, measured values are used. A detailed set of measurements and resulting curve fits have been presented for a subsonic jet by Liu and Maestrello [1]. These were used to evaluate P. Less realistic empirical data were used by Schubert [7] for the same purpose. Then various restrictive assumptions were made in order to reduce Equation (9) into a single equation for the pressure. Numerical solutions were then obtained by both Liu and Maestrello [1] and Schubert [7].

Equation (10) also requires knowledge of the mean flow field in order to evaluate the coefficients of P. In addition, $\bar{\psi}$ must be known in order to evaluate the coefficients of N. Therefore, Equation (9) has to be solved before Equation (10) can be considered. Finally, ϕ' must be known in order to evaluate $N\phi'$. This is especially troublesome, since the fluctuating flow variable ϕ' cannot, in general, be measured directly, and since an analytic or numerical solution for a realistic jet does not at present exist. Therefore, since knowledge of the primitive variable ϕ' is not available, Equation (10) will be modified to relate the correlation functions of ψ' to those of ϕ'. The main thrust of this work is to show how the many spectral properties that can be obtained from experimental data can be used to obtain the scattering of an acoustic wave by a turbulent flow field.

The governing system of equations for $\bar{\psi}$, Equation (9), is elliptic. Since the mean flow field used to evaluate the coefficients is for an axially symmetric jet with $\bar{w} = 0$, the five scalar equations represented by Equation (9) can be reduced to four equations. The solution of the θ momentum equation and the homogeneous boundary conditions on $\bar{\psi}_4$ is $\bar{\psi}_4 = 0$ where $\bar{\psi}_4$ is the component of ψ corresponding to $\tilde{w}$. Then the four remaining equations are independent of θ and the dependent variables are $(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3, \bar{\psi}_5)$ which correspond to $(\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{p})$. The inhomogeneous nature of Equation (10) implies that the component of ψ' corresponding to $\tilde{w}$ will not vanish. However, since all variables are independent of θ, this component will be uncoupled from the other components, again resulting in a system of four equations.

4. Governing equations in terms of correlation

Equation (10) is now considered. This is an elliptic system so that boundary conditions must be specified. They are discussed below.

Let the pair (x, σ) represent a point (x, r) at a time σ, and let the pair (ξ, μ) represent some other point (ξ, η) at some other time μ. Now write Equation (10) for both pairs of coordinates:

$$P_x\psi'(x, \sigma) = -N_x\phi'(x, \sigma), \tag{11}$$

$$P_\xi\psi'(\xi, \mu) = -N_\xi\phi'(\xi, \mu). \tag{12}$$

The subscripts on the operators P and N indicate that differentiations and matrix coefficient evaluations are performed in the coordinates indicated by the subscripts. Now form the product of Equation (11) and the conjugate transpose of Equation (12). The algebraic manipulations are clarified if the operator P is written in more detail. P is a first-order linear differential operator:

$$P_x\psi' = A^k \frac{\partial\psi'}{\partial x_k} + B\psi' \tag{13}$$

where A^k and B are matrices and ψ' a vector and where $x_k = (x, r)$. Then

$$(P_\xi\psi')^* = \frac{\partial\psi'^*}{\partial\xi_l}(A^l)^* + (\psi')^*B^*$$

where ( )* signifies „conjugate transpose". Then the product of the left-hand sides results in:

$$[P_x\psi'(x, \sigma)][P_\xi\psi'(\xi, \mu)]^* = P_x\psi'(x, \sigma)\psi'^*(\xi, \mu)P_\xi^*$$
$$= \left[A^k(x)\frac{\partial}{\partial x_k} + B(x)\right]\left[\psi'(x, \sigma)\frac{\partial\psi'^*}{\partial\xi_l}A^{l*}(\xi) + \psi'(x, \sigma)\psi'^*(\xi, \mu)B^*(\xi)\right]. \tag{14}$$

The operator N is also a first-order linear differential operator:

$$N_x\phi' = C^k \frac{\partial\phi'}{\partial x_k} + D\phi' \tag{15}$$

so that the right-hand side of the above product is given by:

$$[N_x\phi'(x, \sigma)][N_\xi\phi'(\xi, \mu)]^*$$
$$= \left[C^k(x)\frac{\partial}{\partial x_k} + D(x)\right]\left[\phi'(x, \sigma)\frac{\partial\phi'^*}{\partial\xi_l}C^{l*}(\xi) + \phi'(x, \sigma)\phi'^*(\xi, \mu)D^*(\xi)\right]. \tag{16}$$

Since the coordinate pairs ξ_l and x_k are independently chosen,

$$P_x\left\{\frac{\partial}{\partial\xi_l}[\psi'(x, \sigma)\psi'^*(\xi, \mu)]A^{l*}(\xi) + \psi'(x, \sigma)\psi'^*(\xi, \mu)B^*(\xi)\right\}$$
$$= N_x\left\{\frac{\partial}{\partial\xi_l}[\phi'(x, \sigma)\phi'^*(\xi, \mu)]C^{l*}(x) + \phi'(x, \sigma)\phi'^*(\xi, \mu)D^*(\xi)\right\}. \tag{17}$$

Now the meaning of the right-side operators P^* and N^* is clear. They are defined by:

$$yP^* = \frac{\partial y}{\partial x_k}A^{k^*} + yB^* \tag{18}$$

and

$$zN^* = \frac{\partial z}{\partial x_k}C^{k^*} + zD^*. \tag{19}$$

With the above definitions, Equation (17) may be simply written

$$P_x[\psi'(x, \sigma)\psi'^*(\xi, \mu)]P_\xi^* = N_x[\phi'(x, \sigma)\phi'^*(\xi, \mu)]N_\xi^*.$$

Finally, take the time average over the scale of the turbulence. Since the operators P and N are independent of σ:

$$P_x[\overline{\psi'(x, \sigma)\psi'^*(\xi, \mu)}]P_\xi^* = N_x[\overline{\phi'(x, \sigma)\phi'^*(\xi, \mu)}]N_\xi^*. \tag{20}$$

The unknown in Equation (17) is the matrix of correlations $\overline{\psi(x, \sigma)\psi^*(\xi, \mu)}$. The right-hand side is known if the correlation matrix of the fluctuating flow field, $\overline{\phi'(x, \sigma)\phi'^*(\xi, \mu)}$, is known, and if the mean acoustic variable $\bar{\psi}$ is known in order to evaluate N.

5. Boundary conditions

The domain of solution is the whole space (x, r). If the jet exit is taken at $x = 0$, an antijet may be defined for $x < 0$ to avoid the complicated boundary conditions that result from considering the solid surfaces from which the jet emanates. A detailed discussion of the use of an antijet can be found in Schubert [7], and similar concepts were used by Liu and Maestrello [1]. Since the turbulence is confined to the region of the jet, it is reasonable to expect that a similar type of antijet mean flow will serve in the scattering problem.

Boundary conditions for the boundary-value problems (9) and (20) must be imposed at $r = 0$ and at $R = (x^2 + r^2)^{\frac{1}{2}} \to \infty$. At $r = 0$, regularity requires that

$$\tilde{v} = \tilde{w} = 0. \tag{21}$$

For large R, the acoustic variable, $\tilde{\phi}$, is governed by the wave equation, so that $R \to \infty$, $\tilde{\phi}$ must be asymptotic to the solution of the wave equation representing an outgoing spherical wave.

Taking the mean of Equation (21) results in

$$\bar{\psi}_3 = \bar{\psi}_4 = 0, \text{ at } r = 0,$$

so that by using Equation (7)

$$\psi_3' = \psi_4' = 0, \text{ at } r = 0.$$

The acoustic source is placed in the potential core of the jet. If it is desired to avoid the singularity due to the source, the source can be surrounded by a small sphere of radius R_s. In the potential core, the fluctuating flow variable, ϕ', is assumed to vanish. Therefore, if R_s is small enough so that the sphere lies well within the potential core, the mean acoustic variable, $\bar{\psi}$, evaluated at R_s, is given by ψ_s, which is the corresponding variable for a point source in a uniform flow of the same character as that in the potential core. An analytic solution is available for ψ_s (see Moretti and Slutsky [9]). Furthermore, $\phi'(R = R_s) = 0$. Note that ψ_s is axially symmetric and that $w_s = 0$, so that, as stated previously, the boundary conditions for $\bar{\psi}_4$ are homogeneous.

A full description of the boundary conditions for $\bar{\psi}$ is given in Liu and Maestrello [1]. Here it suffices to note that the governing equation for $\bar{\psi}$, Equation (9), is homogeneous while the boundary conditions for $\bar{\psi}$ are inhomogeneous at $R = R_s$. On the other hand, the boundary conditions for ψ' are homogeneous but the governing equation (10) is inhomogeneous.

It is important to note that in solving for $\bar{\psi}$ numerically, the removal of the singular point at the location of the source from the domain of solution is a necessity in order to have finite boundary values. In solving for ψ', the boundary values vanish, and therefore are not singular. However, the singularity is contained within $\bar{\psi}$, which appears in the right-hand side of the equation governing ψ'. Therefore, again the singular point is to be excluded from the domain of solution in order to have a bounded forcing function in Equation (10).

6. Discrete analogs and method of solution

Since Equation (9) has been solved by various authors, it is reasonable to assume that P^{-1}, and therefore $(P^*)^{-1}$ exists. Then, formally, the solution of Equation (20) is

$$\overline{\psi'(x, \sigma)\psi'^*(\xi, \mu)} = (P_x)^{-1} N_x[\overline{\phi'(x, \sigma)\phi'^*(\xi, \mu)}] N_\xi^*(P_\xi^*)^{-1} \tag{22}$$

and, moreover, the limit as $\xi \to x$ of Equation (22) can be evaluated to yield

$$\lim_{\xi \to x} \overline{\psi'(x, \sigma)\psi'^*(\xi, \mu)}$$

whose real part is related to the power spectra of ψ'. The remaining question is how to perform the invertions appearing in Equation (22).

Due to the complex nature of the mean flow field, the coefficient matrices appearing in P are complicated functions of x and r. Therefore, it is improbable that an analytic description of P^{-1} could be obtained. Therefore attention is now turned toward obtaining a discrete approximation to the governing system. Due to the large number of unknowns, a higher order numerical scheme should be used. This may be either a finite-difference or a finite-element approximation. In either case, the discrete approximation to the operator P is a matrix which is denoted by $\mathscr{P}$. In the finite-difference case, $\mathscr{P}$ will operate on discrete values of the unknown variable. If a Galerkin finite-element approach is used in conjunction with cubic B-splines, the discrete operator $\mathscr{P}$ will operate on weight functions from which the unknown variable can be obtained. In either case, inhomogenities in the differential equation or the boundary conditions appear in the discrete system as known inhomogeneous terms. Then, the discrete version of Equation (9) with appropriate boundary conditions can be expressed as

$$\mathscr{P}\underline{\psi} = f. \tag{23}$$

The solution to this discrete system is then

$$\underline{\psi} = \mathscr{P}^{-1}f, \tag{24}$$

so that the discrete analog to P^{-1} is then $\mathscr{P}^{-1}$. Here the reduced operator P is being considered, that is, the θ momentum equation has been uncoupled from the remaining four equations. The inversion of $\mathscr{P}$ and the subsequent evaluation of $\mathscr{P}^{-1}f$ is equivalent to solving the problem considered by Liu and Maestrello [1]. The vector f is related to the boundary conditions. In practice, $\mathscr{P}^{-1}$ would not be calculated, but more efficient means of solving the algebraic system (23) would be employed.

Now consider obtaining a solution to a discrete approximation of Equation (20), that is, a discrete approximation of Equation (22). First, consider the discretization of the operators P_x and N_x. Then

$$\mathscr{P}_x\{\overline{\psi'(x_j, \sigma)\psi'^*(\xi, \mu)}P_\xi\} = \mathscr{N}_x\{\overline{\phi'(x_n, \sigma)\phi'^*(\xi, \mu)}N_\xi^*\}.$$

The operator $\mathscr{P}_x$ is a matrix, as is $\mathscr{N}_x$, the discrete approximation to N_x. The subscripts j and n indicate a grid point of the discretization of the domain of solution. The number of grid points is assumed to be J. The dimension of $\mathscr{P}$ and $\mathscr{N}$ is the product of the number of equations, four, by the number of grid points, J. The operand of $\mathscr{P}_x$ is the matrix which consists of a column of J 4×4 matrices, each of which is the matrix $\overline{\psi'(x_j, \sigma)\psi^*(\xi, \mu)}$ operated on by the differential right-side operator P_ξ^*. Similarly for the operand of $\mathscr{N}_x$. These operands are a function of ξ. Now a discretization in ξ space must be obtained. The result is:

$$\mathscr{P}_x\{\overline{\psi'(x_j, \sigma)\psi'^*(\xi_k, \mu)}\}\mathscr{P}_\xi^* = \mathscr{N}_x\{\overline{\phi'(x_n, \sigma)\phi'^*(\xi_h, \mu)}\}\mathscr{N}_\xi^*. \tag{25}$$

Now all operators and operands are matrices, and operator inverses become simply matrix inverses. The subscripts k and h indicate a grid point of the discretization of the solution domain in ξ coordinates. The turbulent flow correlations need to be known at discrete points x_j with respect to other discrete points ξ_k. The dimensions of $\mathscr{P}$, $\mathscr{N}$, $\overline{\psi'\psi'^*}$, and $\overline{\phi'\phi'^*}$ are $4J$, that is, each is a $4J \times 4J$ complex matrix.

In principle, the solution is now at hand. With the correlation matrix $\overline{\phi'(x_n, \sigma)\phi'^*(\xi_h, \mu)}$ known, the right-hand side of Equation (25) can be evaluated by two matrix multiplications. Then the multiplication on the left by $\mathscr{P}_x^{-1}$, and on the right by $\mathscr{P}_\xi^{*-1}$ results in isolation of the unknown matrix

$$\overline{\psi'(x_j, \sigma)\psi'^*(\xi_k, \mu)}$$

where

$$\overline{\psi'(x_j, \sigma)\psi'^*(\xi_k, \mu)} = \mathscr{P}_x^{-1}\mathscr{N}_x\overline{\phi'(x_n, \sigma)\phi'^*(\xi_h, \mu)}\mathscr{N}_\xi^*(\mathscr{P}_\xi^*)^{-1}, \tag{26}$$

or, in subscript notation

$$\overline{\psi_j'\psi_k'^*} = (\mathscr{P}_{lj})^{-1}\mathscr{N}_{ln}\overline{\phi_n'\phi_h'^*}\mathscr{N}_{mh}^*(\mathscr{P}_{mk}^*)^{-1}, \tag{27}$$

where repeated indices are summed.

Some simplification in the computation of the right-hand side of Equation (27) is possible. If either n or k represents a point outside the jet, then $\overline{\phi_n'\phi_h'^*}$ vanishes since outside the jet there is no turbulence. Furthermore, if n and h represent two points which are far apart, $\overline{\phi_n'\phi_h'^*}$ again vanishes, since distant points are weakly correlated. Finally, many elements of the 4×4 matrix $\overline{\phi_n'\phi_h'^*}$ vanish since different components of ϕ' may be weakly correlated.

In general not all of $\overline{\psi_j'\psi_k'^*}$ will be of interest. If only some elements of $\overline{\psi_j'\psi_k'^*}$ are of interest, great savings can be effected in the computations of Equation (27) since for each desired element of $\overline{\psi_j'\psi_k'^*}$ only one row of $\mathscr{P}_{lj}^{-1}$ and one column of $(\mathscr{P}_{mk}^*)^{-1}$ need be known. This is easy to see if Equation (27) is thought of as the multiplication of matrices. If only element of $\overline{\psi_j'\psi_k'^*}$ is desired, then only that row of $(\mathscr{P}_{lj})^{-1}$ and that column of $\mathscr{N}_{ln}\overline{\phi_n'\phi_h'^*} \times \mathscr{N}_{mh}^*(\mathscr{P}_{mk}^*)^{-1}$ which correspond to the row and column of the desired element need be known. In addition, to know one column of the latter matrix, only the corresponding column of $(\mathscr{P}_{mk}^*)^{-1}$ need be known. Therefore to compute one element of $\overline{\psi_j'\psi_k'^*}$ only one row of $(\mathscr{P}_{lj})^{-1}$ and one column of $(\mathscr{P}_{mk}^*)^{-1}$ need be known. But a column of $(\mathscr{P}_{mk}^*)^{-1}$ is just the complex conjugate of a row of $(\mathscr{P}_{lj})^{-1}$ since $(\mathscr{P}_{mk}^*)^{-1} = (\mathscr{P}_{mk}^{-1})^*$. Therefore, if a diagonal element of $\overline{\psi_j'\psi_k'^*}$ is desired, as probably would be the case, only one row of the matrix inverse need be known. This is, of course, much less expensive to obtain than the whole inverse.

7. Conclusion

This paper provides the basic theoretical analysis and mathematical formulation of the problem of sound propagation through a subsonic turbulent jet. The effects of convection, refraction and scattering are included, and therefore is an extension of previous work in the field.

It is shown that in order to use experimentally measurable quantities, the governing equations for the acoustic variables must be recast in terms of space-time correlations. The equations in terms of correlations are derived. These equations and their discrete analogs are formally inverted to show that the correlations of the acoustic variables depend on the correlations of the turbulent flow variables (and not on the primitive turbulent flow variables themselves) as well as the mean flow field and the mean acoustic variables.

While this work is concerned with subsonic jets, a similar approach may be taken for the propagation of sound through other turbulent flow fields, so long as the relative magnitudes of the mean, turbulent and acoustic fields are similar to those in a turbulent jet.

Appendix 1. The operators *P* and *N*

The operator P is given by:

$$P = i\omega + \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial r} + \alpha_3$$

where

$$\alpha_1 = \begin{pmatrix} \bar{u} & \bar{\rho} & 0 & 0 & 0 \\ 0 & \bar{u} & 0 & 0 & 1/\bar{\rho} \\ 0 & 0 & \bar{u} & 0 & 0 \\ 0 & 0 & 0 & \bar{u} & 0 \\ 0 & \gamma\bar{p} & 0 & 0 & \bar{u} \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} \bar{v} & 0 & \bar{\rho} & 0 & 0 \\ 0 & \bar{v} & 0 & 0 & 0 \\ 0 & 0 & \bar{v} & 0 & 1/\bar{\rho} \\ 0 & 0 & 0 & \bar{v} & 0 \\ 0 & 0 & \gamma\bar{p} & 0 & \bar{v} \end{pmatrix}$$

and

$$\alpha_3 = \begin{pmatrix} (\bar{u}_x + (r\bar{v})_r/r) & \bar{\rho}_x & (r\bar{\rho})_r/r & 0 & 0 \\ -\bar{p}_x/\bar{\rho}^2 & \bar{u}_x & \bar{u}_r & 0 & 0 \\ -\bar{p}_r/\bar{\rho}^2 & \bar{v}_x & \bar{v}_r & 0 & 0 \\ 0 & 0 & 0 & \bar{v}/r & 0 \\ 0 & \bar{p}_x & (r\bar{p})_r/r & 0 & \gamma(\bar{u}_x + (r\bar{v})_r/r) \end{pmatrix}$$

The operator N is given by:

$$N = \beta_1 \frac{\partial}{\partial x} + \beta_2 \frac{\partial}{\partial r} + \beta_3$$

where

$$\beta_1 = \begin{pmatrix} \underline{u} & \rho & 0 & 0 & 0 \\ 0 & \underline{u} & 0 & 0 & -\underline{\rho}/\bar{\rho}^2 \\ 0 & 0 & \underline{u} & 0 & 0 \\ 0 & 0 & 0 & \underline{u} & 0 \\ 0 & \gamma\underline{p} & 0 & 0 & \underline{u} \end{pmatrix} \qquad \beta_2 = \begin{pmatrix} \underline{v} & 0 & \rho & 0 & 0 \\ 0 & \underline{v} & 0 & 0 & 0 \\ 0 & 0 & \underline{v} & 0 & -\underline{\rho}/\bar{\rho}^2 \\ 0 & 0 & 0 & \underline{v} & 0 \\ 0 & 0 & \gamma\underline{p} & 0 & \underline{v} \end{pmatrix}$$

and

$$\beta_3 = \begin{pmatrix} (\underline{u}_x + (r\underline{v})_r/r & \underline{\rho}_x & (r\underline{p})_r/r & 0 & 0 \\ (-\underline{p}_x + \underline{\rho}\bar{p}_x/\bar{\rho})/\bar{\rho}^2 & \underline{u}_x & \underline{u}_r & 0 & 0 \\ (-\underline{p}_r + \underline{\rho}\bar{p}_r/\bar{\rho})/\bar{\rho}^2 & \underline{v}_x & \underline{v}_r & 0 & 0 \\ 0 & 0 & 0 & \underline{v}/r & 0 \\ 0 & \underline{p}_x & (r\underline{p})_r/r & 0 & \gamma(\underline{u}_x + (r\underline{v})_r/r) \end{pmatrix}$$

and where $\underline{\psi} = (\underline{\rho}, \underline{u}, \underline{v}, \underline{w}, \underline{p})$. Subscripts indicate differentiations. Note that $\bar{w} = 0$ and $\underline{w} = 0$, as well as the independence of θ of the problem is built into the above definition.

When ψ is treated as a four-component vector without the component corresponding to w, then the fourth row and fourth column of the above matrices should be omitted.

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